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Cycles of Bott-Samelson type for taut representations

Claudio Gorodski* and Gudlaugur Thorbergsson

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1 Introduction

Ehresmann ([5]) introduced the Schubert varieties of the complex Grassmannians and showed that they give rise to a cell decomposition. Of the many generalizations of his result, the most relevant for our work is the decomposition of the generalized real flag manifolds G/P into Bruhat cells (see [3, 4, 9, 22]). Here G is a noncompact real semisimple Lie group with finite center and P is a parabolic subgroup.

In their study of the Morse theory of symmetric spaces, Bott and Samelson ([2]) came up with concrete cycles in the orbits of their isotropy representations which represent a basis in $\mathbf{Z}_2$-homology. It turns out that those orbits coincide with the generalized real flag manifolds and the images of the cycles of Bott and Samelson equal the closures of the Bruhat cells (see [8] and the appendix to this paper). As an application of the cycles of Bott and Samelson, distance functions to orbits of isotropy representations of symmetric spaces are perfect in the sense that the Morse equalities are in fact equalities. We say that submanifolds with this property are taut and call representations taut if all of their orbits are taut. Because of their tautness, one sees that the Bruhat decomposition of the spaces G/P is minimal in the sense that the number of cells in dimension k is equal to the kth $\mathbf{Z}_2$-Betti number of G/P. Notice that in the case G is a complex group, the cells are all even dimensional which makes the minimality of the Bruhat decomposition trivial, but such easy arguments do not apply otherwise. Another Morse theoretic interpretation of the Bruhat cells is their appearance as the unstable manifolds of suitable height functions on G/P seen as orbits of isotropy representations of symmetric spaces, see [1] and [12].

A construction similar to Bott and Samelson's was used by the second author in [20] to prove the tautness of complete proper Dupin hypersurfaces. Hsiang, Palais and Terng were later able to construct such cycles in isoparametric submanifolds ([11]), and Terng generalized that to weakly isoparametric submanifolds in [18] and to isoparametric submanifolds in Hilbert spaces in [19]. A further application of such constructions can be found in [21] where a necessary topological condition is derived in order for a manifold to admit a taut embedding. In [2] as well as in the other papers, the cycles representing the homology of the submanifold (or orbit) are maps of iterated bundles of curvature surfaces (in general

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spheres) associated to the focal points along geodesics normal to that submanifold. Roughly speaking, as one moves along the normal geodesic towards the submanifold, each focal point accounts for an iteration of the bundle, see Section 2.

We were able to show in [6] that a taut irreducible representation of a compact connected Lie group is either the isotropy representation of a symmetric space or it is one of the following orthogonal representations ($n \geq 2$): the (standard) $\otimes_{\mathbf{R}}$ (spin) representation of $\mathbf{SO}(2) \times \mathbf{Spin}(9)$; or the (standard) $\otimes_{\mathbf{C}}$ (standard) representation of $\mathbf{U}(2) \times \mathbf{Sp}(n)$; or the (standard)$^3 \otimes_{\mathbf{H}}$ (standard) representation of $\mathbf{SU}(2) \times \mathbf{Sp}(n)$. In this paper we will show how to adapt the construction of the cycles of Bott and Samelson to the orbits of these three representations. As a result, they also admit explicit cycles representing a basis of their $\mathbf{Z}_2$-homology and, in particular, this provides another proof of their tautness.

The main new technical difficulty that we encounter in our work is that focal points in the direction of normal vector fields parallel along some curvature circles do not have constant multiplicity, which makes a modification in the construction of the cycles necessary to prevent the bundles from having some degenerate fibers. The modifications required by this "collapsing of focal points" are only needed in finitely many points in one step of the construction and are achieved by a "cut and paste" procedure, see Section 4. This strongly relies on a detailed knowledge of the geometry of the orbits of our representations, which is assembled in Section 3.

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2 The method of Bott and Samelson

Recall that a compact submanifold M of an Euclidean space V is called *taut* with respect to the field of coefficients F if every (squared) distance function of a point $q \in V$, namely $L_q : M \to \mathbf{R}$ given by $L_q(x) = ||q-x||^2$, which is a Morse function, has the minimum number of critical points allowed by the Morse inequalities with respect to F. Throughout this paper we shall be assuming $F = \mathbf{Z}_2$ and dropping the reference to the field.

Now assume that M is a G-orbit of an orthogonal representation $\rho : G \to \mathbf{O}(V)$ of the compact connected Lie group G and fix a G-regular point $q \in V$ such that L_q is a Morse function and therefore has only finitely many critical points on M with pairwise distinct critical values. For each critical point $p \in M$, Bott and Samelson constructed in [2] a compact manifold Γ_p of dimension less than or equal to the index of L_q at p and a smooth map $h_p : \Gamma_p \to M$. Under the assumption of variational completeness (see [2] or [6] for the definition of this concept) they showed that the dimension of Γ_p is equal to the index of L_q at p and that

$$\oplus_p h_{p_*} : \bigoplus_p H_\lambda(\Gamma_p) \to H_\lambda(M)$$

is an isomorphism, where p runs through all the critical points of L_q on M such that the index of L_q at p equals λ. This implies that the Morse inequalities for L_q are equalities, i. e. M is taut.

In fact, for a real number c set

$$M^{c-} = \{x \in M : L_q(x) < c\} \quad \text{and} \quad M^c = \{x \in M : L_q(x) \leq c\}.$$

We know that M^c has the same homotopy type as M^{c-} unless c is a critical value of L_q. Assume this is the case and let p be the corresponding critical point, $L_q(p) = c$. Then M^c has the homotopy type of M^{c-} with a λ-cell e_λ attached, where λ is the index of L_q at p. Consider the homology sequence of the pair (M^c, M^{c-}):

$$\begin{aligned} \ldots \to \underbrace{H_{\lambda+1}(M^c, M^{c-})}_{=0} \to H_\lambda(M^{c-}) \to H_\lambda(M^c) \\ \to \underbrace{H_\lambda(M^c, M^{c-})}_{=\mathbf{Z}_2} \xrightarrow{\partial_*} H_{\lambda-1}(M^{c-}) \to H_{\lambda-1}(M^c) \to \underbrace{H_{\lambda-1}(M^c, M^{c-})}_{=0} \to \ldots \end{aligned}$$

By passing from M^{c-} to M^c the only possible changes in homology occur in dimensions $\lambda - 1$ and λ. In the first case, the boundary ∂e_λ of the attaching cell is a $(\lambda-1)$-sphere in M^{c-} that does not bound a chain in M^{c-}, so e_λ has as boundary the nontrivial cycle ∂e_λ in M^{c-} and the map ∂_* is not zero. In the second case, ∂e_λ does bound a chain in M^{c-}, which we cap with e_λ to create a new nontrivial homology class in M^c, so ∂_* is zero and $H_\lambda(M^c) \cong H_\lambda(M^{c-}) \oplus \mathbf{Z}_2$. Thus the Morse inequalities are equalities when

$$H_\lambda(M^c) \to H_\lambda(M^c, M^{c-1}) \tag{2.1}$$

is surjective, where c is an arbitrary critical value of L_q and λ is the index of L_q at the corresponding critical point.

We proceed to describe the construction of Bott and Samelson that yields the surjectivity of (2.1) under the assumption of variational completeness. So fix c, p, λ as above, let $f_1, \ldots, f_r$ be the focal points of M on the segment $\overline{qp}$ in focal distance decreasing order and let $m_1, \ldots, m_r$ be their respective multiplicities. Note that $H = G_q$ is a principal isotropy group. We have that $\overline{qp}$ is perpendicular to M at p, therefore it is also perpendicular to Gq at q, so that H fixes the segment $\overline{qp}$ pointwise. Next let the r-fold product H^r act on the product manifold $G_{f_1} \times \ldots \times G_{f_r}$ by the rule

$$\mathbf{g} \cdot \mathbf{h} = (g_1 h_1,\, h_1^{-1} g_2 h_2,\, h_2^{-1} g_3 h_3,\, \ldots\, ,\, h_{r-1}^{-1} g_r h_r),$$

where $\mathbf{g} = (g_1, \ldots, g_r) \in G_{f_1} \times \ldots \times G_{f_r}$ and $\mathbf{h} = (h_1, \ldots, h_r) \in H^r$. Let Γ_p be the quotient manifold under this action, namely, $\Gamma_p = G_{f_1} \times_H G_{f_2} \times_H \ldots \times_H G_{f_r}/H$, and define $h_p : \Gamma_p \to M$ by $h[(g_1, \ldots, g_r)] = g_1 \ldots g_r p$. It is immediate that h_p is well-defined. Notice that Γ_p is the total space of an iterated fiber bundle that can be identified with the space of polygonal paths from p to $g_1 \ldots g_r p$ with vertices $g_r p$, $g_{r-1} g_r p$, $\ldots$, $g_2 \ldots g_{r-1} g_r p$ for $(g_1, \ldots, g_r) \in G_{f_1} \times \ldots \times G_{f_r}$. We compute:

$$\begin{aligned} \dim \Gamma_p &= (\dim G_{f_1} - \dim H) + \cdots + (\dim G_{f_r} - \dim H) \\ &= m_1 + \cdots + m_r \qquad \text{(by variational completeness, see Lemma 3.2 in [6])} \\ &= \lambda \qquad \text{(by the index theorem of Morse).} \end{aligned}$$

Since Γ_p is a compact manifold of dimension λ, it follows that $H_\lambda(\Gamma_p) = \mathbf{Z}_2$. Moreover, it is easy to see that $p \in h_p(\Gamma_p) \subset M^{c-} \cup \{p\}$ and that h_p is an immersion near $(1, \ldots, 1) \in G_{f_1} \times \ldots \times G_{f_r}$. Now, locally in a Morse chart centered at p, the image $h_p(\Gamma_p)$ is transversal to the ascending cell so that we can deform it into the descending cell e_λ. Therefore $h_{p_*} : H_\lambda(\Gamma_p) \to H_\lambda(M^c, M^{c-})$ is surjective. Finally, factorize

$$H_\lambda(\Gamma_p) \to H_\lambda(M^c) \to H_\lambda(M^c, M^{c-})$$

to get the surjectivity of (2.1).

Let now M be an arbitrary submanifold of an Euclidean space. A *curvature surface* of M is a submanifold N such that there exists a parallel normal vector field ξ of M along N such that the tangent space $T_r N$ is a full eigenspace of the Weingarten operator $A_{\xi(r)}$ for all $r \in N$.

Turning back to the case of the G-orbit M, it follows easily from variational completeness that for each focal point f_i we have that $G_{f_i} p \subset M$ is a curvature surface of M, so Γ_p can be thought of as an iterated bundle of curvature surfaces. Those curvature surfaces will be spheres if q belongs to an open and dense subset of V.

The cycle constructions in [11, 18, 19, 20, 21] are all based on the observation that one does not need the action of the group G to define the iterated fiber bundle Γ_p but only the curvature surfaces themselves, which then must be assumed to be mutually diffeomorphic in each step to guarantee that one indeed gets a fiber bundle. This follows in all those papers since the multiplicities of the eigenvalues of the Weingarten operator A_ξ are constant for parallel normal vector fields ξ of M along curvature surfaces. However, that assumption is false for the three exceptional taut irreducible representations as we will see at the end of the next section.

3 The reduction principle

Let $\rho : G \to \mathbf{O}(V)$ be an orthogonal representation of a compact Lie group G which is not assumed to be connected. Denote by H a fixed principal isotropy subgroup of the G-action on V and let V^H be the subspace of V that is left pointwise fixed by the action of H. Let N be the normalizer of H in G. Then the group N/H acts on V^H with trivial principal isotropy subgroup. Moreover, the following result is known ([7, 13, 14, 15, 16, 17]):

3.1 Theorem (Luna-Richardson) *The inclusion $V^H \to V$ induces a stratification preserving homeomorphism between orbit spaces*

$$V^H/N \to V/G. \tag{3.2}$$

The injectivity of the map (3.2) means that $Np = Gp \cap V^H$ for $p \in V^H$. In particular, the H-fixed point set of a G-orbit is a smooth manifold.

Observe also that for a regular point $p \in V^H$ the normal space to the principal orbit $M = Gp$ at p is contained in V^H, because the slice representation at p is trivial. More generally, we have:

3.3 Lemma ([6], Lemma 3.17) *Let $p, q \in V^H$ and suppose that q is a regular point for G. Consider $M = Gp$ and let $L_q : M \to \mathbf{R}$ be the distance function. Then the critical set of L_q is contained in $M^H = M \cap V^H$, namely, Crit(L_q) = Crit($L_q|_{M^H}$).*

For $p \in V$ define $\nu_p = T_p(Gp)^\perp$ and for $p \in V^H$ define $\sigma_p = T_p(Np)^\perp \cap V^H$. As a consequence of Lemma 3.3 we immediately get:

3.4 Lemma *For $p \in V^H$, we have that $\sigma_p = \nu_p \cap V^H$.*

We also have:

3.5 Lemma ([6], Lemma 3.16) *For $p \in V^H$, $n \in \sigma_p$ and $M = Gp$ we have that the Weingarten operators satisfy $A_n^M|_{T_pM^H} = A_n^{M^H}$.*

Next fix $p \in V^H$ and consider the isotropy subgroup N_p. Then N_p acts linearly on σ_p. It is easy to apply the reduction principle as in Theorem 3.1 to see that:

3.6 Lemma *The inclusion $\sigma_p \to \nu_p$ induces a stratification preserving homeomorphism between orbit spaces*

$$\sigma_p/N_p \to \nu_p/G_p.$$

In the following we specialize to the case where $\rho : G \to \mathbf{O}(V)$ is one of the three nonpolar taut irreducible representations, namely given by the following table ($n \geq 2$; the last column will be explained later in this section):

case	G	ρ	$\dim V = m$	*D-type*
1	$\mathbf{SO}(2) \times \mathbf{Spin}(9)$	(standard) $\otimes_{\mathbf{R}}$ (spin)	32	$\overset{1}{\circ}\quad \overset{6}{\circ}\quad \overset{7}{\circ}$
2	$\mathbf{U}(2) \times \mathbf{Sp}(n)$	(standard) $\otimes_{\mathbf{C}}$ (standard)	$8n$	$\overset{1}{\circ}\quad \overset{2}{\circ}\quad \overset{4n-5}{\circ}$
3	$\mathbf{SU}(2) \times \mathbf{Sp}(n)$	$(\text{standard})^3 \otimes_{\mathbf{H}}$ (standard)	$8n$	$\overset{4n-5}{\circ}\quad \overset{1}{\circ}\!-\!\!-\!\overset{1}{\circ}$

Then the cohomogeneity of ρ is 3, the dimension of V^H is 4 and N/H consists of finitely many connected components each of which is a circle. In particular the identity component $(N/H)^0$ is a circle group. Set $T = (N/H)^0$. Then N/H is a semidirect product $D \ltimes T$, where D is a discrete group.

Since G acts linearly on V, it is enough to consider its action on the unit sphere $S^{m-1} \subset V$. Henceforth we view the orbits as submanifolds of S^{m-1}. The unit sphere $S^3 \subset V^H$ is totally geodesically embedded in S^{m-1} as the the fixed point set $(S^{m-1})^H$.

The reduced representation of $T \cong \mathbf{SO}(2)$ on $V^H \cong \mathbf{R}^4$ restricts to the Hopf action of T on the unit sphere $S^3 \subset V^H$. For each $p \in S^3$, we shall call the great circle $Tp \subset Gp$ a *special circle*. More generally, we shall call any G-translate of a special circle also a special circle. Notice that through each point $p \in S^{m-1}$ there exists a unique special circle, and that S^3 intersects every G-orbit in a finite number of circles, each of which is a special circle.

An important consequence of the reduction principle for us is the technique which we call "reduction of focal data", namely all the focal information about the G-orbits can be

read off the geometry of S^3. In fact, let $q \in S^{m-1}$ be a focal point of an orbit Gp. We can conjugate by an element of G and then assume that q is a focal point of Gp relative to p and that $p \in S^3$. Moreover, it follows from Lemma 3.6 that

3.7 Lemma *Let $q \in S^{m-1}$ be a focal point of Gp relative to $p \in S^3$. Then q is G_p-conjugate to an element in S^3. In particular, if p is a regular point then $q \in S^3$.*

Next we discuss the multiplicities of the focal points of the orbits and distinguish between three different types of focal points. It is useful to introduce the circle bundle

$$\begin{array}{ccc} T \approx S^1 & \to & S^3 \subset V^H \\ & & \eta \downarrow \\ & & S^2 \approx \mathbf{C}P^1 \end{array} \tag{3.8}$$

We equip the base space S^2 with the quotient metric. Then S^2 is a metric sphere of radius $1/2$ and the discrete group D acts there by isometries. The D-singular set is a union of great circles, and each D-singular great circle C comes with a multiplicity, namely the difference between the dimension of a principal G-orbit and the dimension of the orbit Gq where q is a point in $\eta^{-1}(C)$ not contained in the preimage of any other D-singular great circle. We write the multiplicity of C next to the vertex of the diagram of D which corresponds to C (or to a D-conjugate of C), see the last column in the table.

For fixed $p \in S^3$ and $n \in \sigma_p \cap T_pS^3$ a unit vector, let $\gamma_t(s) = (\cos s)tp + (\sin s)n$, where $s \in \mathbf{R}$, $t \in T$, be the normal geodesic to Gp in the direction of $n \in \sigma_{tp}$. Notice that γ_t is an horizontal curve with respect to the Riemannian submersion η and that n as a point in S^3 satisfies $n \in Gp$ (because $\eta(n) = -\eta(p)$ is D-conjugate to $\eta(p)$).

If $p \in S^{m-1}$ and q is a focal point of Gp relative to p of multiplicity $k > 0$, we shall call q a focal point of *standard type* (resp. *mixed type*, *special type*) if $k = \dim G_q p$ (resp. $k > \dim G_q p > 0$, $\dim G_q p = 0$). Any focal point falls into one of these three cases. Notice that the first case occurs precisely when the focal point satisfies the condition in the definition of variational completeness.

3.9 Proposition *With the above notation:*

a. The orbit spaces $S^{m-1}/G \cong S^2/D$.

b. If $p \in S^3$ is a regular point, then the focal points of standard type of Gp along the normal geodesic segment $\gamma_1|_{[0,\pi)}$ are precisely the points $\gamma_1(s)$, $0 < s < \pi$, such that $\eta(\gamma_1(s))$ belongs to a D-singular great circle C in S^2. The multiplicity of a standard focal point $\gamma_1(s)$ is the sum of the multiplicities of the D-singular great circles which pass through $\eta(\gamma_1(s))$. There is precisely one focal point of special type along $\gamma_1|_{[0,\pi)}$, namely $n \in S^3$, its multiplicity is one, its focal distance is $\pi/2$ and the associated line of curvature is Tp. There are no focal points of mixed type.

c. If $p \in S^3$ is a singular point and $d\eta_p(n)$ is not tangent to a D-singular great circle in S^2, then the focal points of standard type of Gp along the normal geodesic segment

$\gamma_1|_{[0,\pi)}$ are precisely the points $\gamma_1(s)$, $0 < s < \pi$ and $s \neq \pi/2$, such that $\eta(\gamma_1(s))$ belongs to a D-singular great circle C in S^2. The multiplicity of a standard focal point $\gamma_1(s)$ is the sum of the multiplicities of the D-singular great circles which pass through $\eta(\gamma_1(s))$. There is precisely one focal point of mixed type along along $\gamma_1|_{[0,\pi)}$, namely $n \in S^3$, its multiplicity is $\dim G_n p + 1$ and its focal distance is $\pi/2$. There are no focal points of special type.

Proof. Theorem 3.1 and (3.8) imply that $S^{m-1}/G \cong S^3/N \cong S^2/D$. This gives (a). If p is a regular point, then each point $\gamma_1(s)$, $0 < s < \pi$, such that $\eta(\gamma_1(s))$ belongs to a D-singular great circle C in S^2 is a focal point of Gp of multiplicity greater than or equal to the sum of multiplicities of the D-singular great circles which pass through $\eta(\gamma_1(s))$. It follows from a counting argument based on data from the table that the sum of the multiplicities of the focal points of Gp along the normal geodesic segment $\gamma_1|_{[0,\pi)}$ obtained in this way is greater than or equal to $\dim Gp - 1$, none of which has a focal distance equal to $\pi/2$. Now $n \in \sigma_{tp}$ for all $t \in T$, so $n \in S^3$ is a focal point of Gp with focal distance $\pi/2$. Since the sum of the multiplicities of the focal points to Gp along the geodesic segment $\gamma_1|_{[0,\pi)}$ must be equal to $\dim Gp$, we have that there are no other focal points and (b) follows. If p is a singular point, we observe: n is also a singular point; an element in G_n that fixes p must also fix the geodesic segment between n and p. Now the proof of (c) is similar to the above. □

In the cycle constructions in [11, 18, 19, 20, 21] and implicitly in [2], it is essential that the multiplicities of the eigenvalues of A_ξ be constant for ξ a parallel normal vector field along a curvature surface; see the end of Section 2. We will now see that we do not have such constancy for certain parallel normal vectors along the special circles. Notice that the nonconstancy of multiplicities of eigenvalues of A_ξ is equivalent to the nonconstancy of multiplicities of focal points in the direction of ξ, which in turn is equivalent to saying that there is collapsing of focal points in the direction of ξ.

In fact, the differential $d\eta_p : \sigma_p \cap T_p S^3 \to T_{\eta(p)} S^2$ is an isometry and $\tau(t) = d\eta_{tp} \circ P_t \circ d\eta_p^{-1}$ for $t \in T$ defines the standard representation $\tau : T \cong \mathbf{SO}(2) \to T_{\eta(p)} S^2 \cong \mathbf{R}^2$, where P_t denotes parallel transport along the special circle from p to tp. Moreover,

$$\eta(\gamma_t(s)) = (\cos s)\eta(p) + (\sin s)\tau(t) d\eta_p(n),$$

so the distribution of focal points of standard type on the geodesic segment $\gamma_t|_{[0,\pi/2]}$ from tp to n depends on $t \in T$, but is well controlled by the intersections of $\eta \circ \gamma_t|_{[0,\pi/2]}$ with the D-singular set in S^2. In particular, for a finite number of values $t \in T$ there is collapsing of focal points in the direction of n viewed as a parallel normal vector field along the special circle Tp.

The existence of focal points other than of standard type precludes variational completeness (see Lemma 3.1 in [6]). Therefore:

3.10 Theorem ([6]) *The representations listed in the table are not variationally complete.*

4 The construction of the cycles

According to the discussion in the previous section, the representations listed in the table there fail to be variationally complete just because the Jacobi field J along γ_1 defined by

$J(s) = (\cos s)v$, where v is a vector tangent to the special circle Tp at p, is not the restriction of a Killing vector field induced by the representation. The "variational cocompleteness" of the principal orbits of these representations in the sense of [10] is one, and in this regard they are very close to being variationally complete. In this section we want to adapt the construction described in Section 2 to obtain cycles of Bott-Samelson type for the orbits of these nonvariationally complete representations.

So let M be a G-orbit of one of the representations listed in the table, fix a G-regular point $q \in V$ such that the distance function $L_q : M \to \mathbf{R}$ is a Morse function and let $p \in M$ be a critical point of L_q such that $L_q(p) = c$ and the index of L_q at p is λ. We want to construct a compact λ-dimensional manifold Γ_p and a smooth map $h_p : \Gamma_p \to M$ such that $p \in h_p(\Gamma_p) \subset M^{c-} \cup \{p\}$ and h_p is an immersion near $h_p^{-1}(p)$. As in Section 2, this will imply the surjectivity of (2.1) and therefore the tautness of M.

Before we begin, we introduce the following notation for future reference. If $q_1, \ldots, q_u \in S^3$, we denote by $W_{q_1,\ldots,q_u}$ the quotient manifold $G_{q_1} \times_H G_{q_2} \times_H \ldots \times_H G_{q_u}/H$, see Section 2.

We start the construction with the case where M is a principal orbit. If all the focal points to M along the normal geodesic segment $\overline{qp}$ are of standard type, we follow the same procedure as in Section 2 in order to define Γ_p and h_p.

Consider next the case where the furthermost focal point, say f_1, is of special type. Then all the other focal points on the normal geodesic segment are of standard type. It follows from Proposition 3.9 and the discussion in Section 3 that there is a finite subset $\{t_0, \ldots, t_{l-1}\} \subset T$ such that the multiplicities of the focal points to M along the geodesic segment $\overline{q(tp)}$ are locally constant for $t \in T \setminus \{t_0, \ldots, t_{l-1}\}$. Moreover, we may perturb q if necessary in order to have $t_i \neq 1$ for $i = 0, \ldots, l-1$. Note that for all $t \in T \setminus \{t_0, \ldots, t_{l-1}\}$ we can specify the focal points along $\overline{q(tp)}$ as $f_1, f_2(t), \ldots, f_r(t)$ in focal distance decreasing order. Next we assume the t_i's are ordered so that they partition the circle T in closed arcs $[t_0, t_1], [t_1, t_2], \ldots, [t_{l-2}, t_{l-1}], [t_{l-1}, t_0]$. Fix $[t_{i-1}, t_i]$ (indices i modulo l) and define a fiber bundle $\Gamma_p^i \to [t_{i-1}, t_i]$ as follows. For $t \in (t_{i-1}, t_i)$, the isotropy subgroups $G_{f_j(t)}$ for $j = 2, \ldots, r$ are well-defined and we can take the fiber $\Gamma_p^i|_t$ of Γ_p^i over $t \in (t_{i-1}, t_i)$ to be $W_{f_2(t),\ldots,f_r(t)}$. We want to define $G_{f_j(t_{i-1}^+)}$ and $G_{f_j(t_i^-)}$ for $j = 2, \ldots, r$. Note that as $t \to t_{i-1}^+$ (resp. $t \to t_i^-$) two or more of the focal points of M relative to tp collapse into a focal point relative to $t_{i-1}p$ (resp. $t_i p$) whose multiplicity is equal to the sum of the multiplicities of the collapsing focal points. If j is an index for which the focal point $f_j(t)$ is not collapsing as $t \to t_{i-1}^+$ (resp. $t \to t_i^-$) we simply take $G_{f_j(t_{i-1}^+)}$ (resp. $G_{f_j(t_i^-)}$) to be the isotropy subgroup $G_{f_j(t_{i-1})}$ (resp. $G_{f_j(t_i)}$). Otherwise, suppose that j is an index for which $f_j(t)$ collapses as $t \to t_{i-1}^+$ (resp. $t \to t_i^-$) into a focal point $\bar{f}$. Then $\eta(f_j(t))$ belongs to a D-singular great circle C which also contains $\eta(\bar{f})$ (here $t \in (t_{i-1}, t_i)$). Define the subgroup $G_{\tilde{C}} \subset G_{\bar{f}}$ to be the stabilizer of the η-horizontal lift of C through $\bar{f}$, which we call $\tilde{C}$. Alternatively, $G_{\tilde{C}}$ could be defined as a certain isotropy subgroup relative to the slice representation at $\bar{f}$. Since $t \mapsto f_j(t)$ and $\tilde{C}$ are both smooth lifts of C for $t \in (t_{i-1}, t_i)$, it follows that there exists a smooth curve $n_j(t) \in T$ such that $G_{f_j(t)} = n_j(t) G_{\tilde{C}} n_j^{-1}(t)$ for $t \in (t_{i-1}, t_i)$. Moreover, as $t \to t_i^+$ (resp. $t \to t_{i+1}^-$) we have that $f_j(t) \to \bar{f}$, so $\lim_{t \to t_{i-1}^+} n_j(t) = 1$ (resp. $\lim_{t \to t_i^-} n_j(t) = 1$) and n_j can be smoothly extended to $[t_{i-1}, t_i)$ (resp. $(t_{i-1}, t_i]$). Let $G_{f_j(t_{i-1}^+)} = G_{\tilde{C}}$ (resp. $G_{f_j(t_i^-)} = G_{\tilde{C}}$). Now we can define the fiber $\Gamma_p^i|_{t_{i-1}}$ (resp. $\Gamma_p^i|_{t_i}$) of Γ_p^i over t_{i-1} (resp. t_i) to be $W_{f_2(t_{i-1}^+),\ldots,f_r(t_{i-1}^+)}$,

(resp. $W_{f_2(t_i^-),\dots,f_r(t_i^-)}$). The previous discussion shows that Γ_p^i is a smooth fiber bundle over $[t_{i-1}, t_i]$. We also define a smooth map $h_p^i : \Gamma_p^i \to M$ by $h_p^i[(g_2, \dots, g_r)] = g_2 \dots g_r p$, where $(g_2, \dots, g_r) \in \Gamma_p^i|_t$.

For each $i : 1, \dots, l$, note that $h_p^i(\Gamma_p^i|_{t_i}) = h_p^{i+1}(\Gamma_p^{i+1}|_{t_i})$ and denote this subset of M by A_i. The last step in the construction is the definition of a 'glueing' diffeomorphism $\psi_i : \Gamma_p^i|_{t_i} \to \Gamma_p^{i+1}|_{t_i}$ and a 'correcting' diffeomorphism $\phi_i : M \to M$ for each $i = 1, \dots, l$ such that $\phi_{i+1}(A_i) = A_i$ and

$$h_p^i|_{\Gamma_p^i|_{t_i}} = \phi_{i+1} h_p^{i+1} \psi_i \tag{4.1}$$

for $i = 1, \dots, l$ (indices i modulo l). We also require the 'cycle condition'

$$\phi_1 \phi_2 \cdots \phi_l = 1. \tag{4.2}$$

Then we will be able to define Γ_p as the resulting fiber bundle over S^1 and $h_p : \Gamma_p \to M$ as the map that restricts to $\phi_2 \phi_3 \cdots \phi_i h_p^i$ on each closed subset Γ_p^i.

According to the discussion in Section 3, for each t_i we have either one collapsing of two focal points (*double collapse*; occurs six times for each representation listed in the table) or one collapsing of three focal points (*triple collapse*; occurs twice for the representation listed in the table under number 3). We discuss these two possibilities separately.

Double collapse

Suppose that for some $j = 2, \dots, r-1$ we have that $f_j(t)$ and $f_{j+1}(t)$ collapse together into a focal point $\bar{f}$ as $t \to t_i$. In this case we have that $G_{f_j(t_i^+)} = G_{f_{j+1}(t_i^-)}$ and $G_{f_{j+1}(t_i^+)} = G_{f_j(t_i^-)}$. It is not difficult to see that $G_{f_j(t_i^+)} \cap G_{f_{j+1}(t_i^+)} = H$ and that $(G_{f_j(t_i^+)}, G_{f_{j+1}(t_i^+)})$ is a factorization of the isotropy subgroup $G_{\bar{f}}$, that is $G_{\bar{f}} = G_{f_j(t_i^+)} \cdot G_{f_{j+1}(t_i^+)}$. It follows that the multiplication map $G_{f_j(t_i^+)} \times G_{f_{j+1}(t_i^+)} \to G_{\bar{f}}$ induces a diffeomorphism

$$G_{f_j(t_i^+)} \times_H G_{f_{j+1}(t_i^+)} \approx G_{\bar{f}}.$$

Clearly there is a similar diffeomorphism $G_{f_j(t_i^-)} \times_H G_{f_{j+1}(t_i^-)} \approx G_{\bar{f}}$. Now the composed diffeomorphism $G_{f_j(t_i^-)} \times_H G_{f_{j+1}(t_i^-)} \to G_{f_j(t_i^+)} \times_H G_{f_{j+1}(t_i^+)}$ extends trivially to a diffeomorphism

$$\psi_i : W_{f_2(t_i),\dots,f_j(t_i^-),f_{j+1}(t_i^-),\dots,f_r(t_i)} \to W_{f_2(t_i),\dots,f_j(t_i^+),f_{j+1}(t_i^+),\dots,f_r(t_i)}$$

which satisfies (4.1) if we take $\phi_{i+1} = 1$. Observe that this choice of ϕ_{i+1} leaves unaffected the cycle condition (4.2).

Triple collapse

Suppose that for some $j = 2, \dots, r-2$ we have that $f_j(t)$, $f_{j+1}(t)$ and $f_{j+2}(t)$ collapse together into a focal point $\bar{f}$ as $t \to t_i$. In this case we have that $G_{f_j(t_i^+)} = G_{f_{j+2}(t_i^-)}$, $G_{f_{j+1}(t_i^+)} = G_{f_{j+1}(t_i^-)}$ and $G_{f_{j+2}(t_i^+)} = G_{f_j(t_i^-)}$. This is the case of representation number 3 in the table, so the group D is of $A_1 \times A_2$-type, namely D is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{D}_3$ where $\mathbf{D}_3$ is the dihedral group of degree 3. Note that the N-isotropy subgroup at $\bar{f}$ equals $\mathbf{D}_3$, and

let $w^\circ \in \mathbf{D}_3$ be the unique element of order 2 which conjugates $G_{f_j(t_i^+)}$ into $G_{f_{j+2}(t_i^+)}$. Then w° normalizes $G_{f_{j+1}(t_i^+)}$ and H, so the diffeomorphism

$$G_{f_j(t_i^-)} \times_H G_{f_{j+1}(t_i^-)} \times_H G_{f_{j+2}(t_i^-)} \to G_{f_j(t_i^+)} \times_H G_{f_{j+1}(t_i^+)} \times_H G_{f_{j+2}(t_i^+)}$$

given by

$$(g_1, g_2, g_3) \mapsto (w^\circ g_1 w^\circ, w^\circ g_2 w^\circ, w^\circ g_3 w^\circ)$$

is well-defined and extends trivially to a diffeomorphism

$$\psi_i : W_{f_2(t_i),\dots,f_j(t_i^-),f_{j+1}(t_i^-),f_{j+2}(t_i^-),\dots,f_r(t_i)} \to W_{f_2(t_i),\dots,f_j(t_i^+),f_{j+1}(t_i^+),f_{j+2}(t_i^+),\dots,f_r(t_i)}$$

which satisfies (4.1) if we take ϕ_{i+1} to be conjugation by w°. Observe that the triple collapse occurs precisely twice for this representation, and it is easy to see that in both instances the element w° is the same. Since w° has order two, condition (4.2) is also satisfied.

The above discussion finishes the construction for the case where the furthermost focal point on the normal geodesic segment $\overline{qp}$ is of special type. We now consider the case where there is a focal point of special type on that segment which is not the furthermost focal point, say it is f_r, the r-th focal point in focal distance decreasing order for $r \geq 2$. Let f_{r-1} be the previous focal point and choose an interior point in the segment $\overline{f_{r-1}f_r}$ to be denoted q'. We have that the furthermost focal point to M along the normal geodesic segment $\overline{q'p}$ is f_r, hence of special type, so we may apply the previous case and obtain a cycle $h_p^0 : \Gamma_p^0 \to M$ for this situation. Note that H acts naturally on the left on the manifold Γ_p^0 (this is just left multiplication for representations number 1 and 2; for representation number 3, note that the two triple collapses correspond to points in the special circle which cut it into two halves, and that the element $h \in H$ acts by left multiplication by h on the fibers lying over one half of the special circle and acts by left multiplication by $w^\circ h w^\circ$ on the fibers lying over the other half). The $r-1$ first focal points $f_1, \dots, f_{r-1}$ along $\overline{qp}$ are all of standard type. We define $\Gamma_p = G_{f_1} \times_H \dots \times G_{f_{r-1}} \times_H \Gamma_p^0$ and $h_p : \Gamma_p \to M$ by $h_p[(g_1, \dots, g_{r-1}, \mathbf{g}_0)] = g_1 \dots g_{r-1} h_p^0(\mathbf{g}_0)$, where $(g_1, \dots, g_{r-1}) \in G_{f_1} \times \dots \times G_{f_{r-1}}$, $\mathbf{g}_0 \in \Gamma_p^0$. This concludes the discussion for the case of a principal orbit M.

In the case where M is not a principal orbit, the procedure is similar. We just remark that when the furthermost focal point along the normal geodesic segment $\overline{qp}$, say f_1, is of mixed type, then $f_1 = n$ and one does "cut and paste" to construct Γ_p as a fiber bundle over S^1 using typical fibers of the form $W_{f_1(t),f_2(t),\dots,f_s(t)}$. This completes the construction of the cycles of Bott-Samelson type for the exceptional taut irreducible representations.

Questions 1. Are there orthogonal representations which have principal orbits with "variational cocompleteness" one (see [10]) without being taut?
2. Is it possible to generalize the construction given in this paper and find explicit cycles for the $\mathbf{Z}_2$-homology of an arbitrary taut submanifold M of an Euclidean space?
3. Assume the answer to the question in (2) to be positive, say for some subclass of taut submanifolds at least. Is it then possible to give an upper bound for the number of cycles and hence estimate the $\mathbf{Z}_2$-Betti numbers of M? One might conjecture the upper bound for the kth Betti number to be $\binom{k}{n}$ where n is the dimension of M.

Appendix: The Bruhat cells versus the Bott-Samelson cycles

In this appendix we prove that the closures of the Bruhat cells coincide with the images of the Bott-Samelson cycles in a real generalized flag manifold G/P. We are not aware of any such proof in the literature, but the special case where G is a complex group and P is a Borel subgroup is treated by Hansen in [8]. Our proof is analogous to his.

We follow closely (but not strictly) the setting and notation from Sections 1.1 and 1.2 in [22]. Let G be a noncompact real connected semisimple Lie group with finite center and denote by $\mathfrak{g}$ its Lie algebra. Consider a fixed Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with corresponding Cartan involution θ and equip $\mathfrak{g}$ with the real inner product $(X, Y)_\theta = -B(X, \theta Y)$ where $X, Y \in \mathfrak{g}$ and B is the Killing form of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal Abelian subspace in $\mathfrak{p}$. Then we have the real orthogonal restricted root decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \sum_{\lambda\in\Sigma} \mathfrak{g}_\lambda$ where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Introduce an order in the dual $\mathfrak{a}^*$ and let Σ^+ (resp. Σ^-) denote the set of positive (resp. negative) restricted roots. Then $\mathfrak{n}^+ = \sum_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$ and $\mathfrak{n}^- = \theta(\mathfrak{n}^+) = \sum_{\lambda\in\Sigma^-} \mathfrak{g}_\lambda$ are nilpotent subalgebras of $\mathfrak{g}$. Let K, A, N^+, N^- be respectively the analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}^+$ and $\mathfrak{n}^-$. Then K, A, N^+ and N^- are closed subgroups of G and we have the Iwasawa decomposition $G = KAN^+$. Let M be the centralizer of A in K and let M^* be the normalizer of A in K. Then the Weyl group of $(\mathfrak{g}, \mathfrak{a})$ generated by reflections on the singular hyperplanes in $\mathfrak{a}$ (which are the kernels of the reduced restricted roots) is identified with M^*/M and $B = MAN^+$ is a Borel subgroup of G. Define also $U_w^+ = N^+ \cap wN^-w^{-1}$ where $w \in W$ (we abuse the notation in the evident manner). Then we have (see [3, 4, 9, 22]):

A.1 Theorem *The quotient manifold G/B is the disjoint union of the N^+-orbits $N^+w^{-1}B$ for $w \in W$. Moreover, for each $w \in W$, the map $U_{w^{-1}}^+ \to N^+w^{-1}B$, $v \mapsto vw^{-1}B$ is a diffeomorphism.*

Theorem A.1 admits the following generalization. Let $\Upsilon \subset \Sigma^+$ be the system of simple roots. The cardinality l of Υ is equal to the dimension of $\mathfrak{a}$ and l is the split-rank of $\mathfrak{g}$. Fix a subset Θ of Υ and denote by $<\Theta>$ the set comprised of those λ in Σ which are linear combinations of the elements of Θ; we also agree to write $<\Theta>^\pm$ for $\Sigma^\pm \cap <\Theta>$. Put $\mathfrak{n}^\pm(\Theta) = \sum_\lambda \mathfrak{g}_\lambda$, $\lambda \in <\Theta>^\pm$ and $\mathfrak{n}_\Theta^+ = \sum_\lambda \mathfrak{g}_\lambda$, $\lambda \in \Sigma^+ \setminus <\Theta>^+$; $\mathfrak{n}_\Theta^- = \theta(\mathfrak{n}_\Theta^+)$. Put also $\mathfrak{a}(\Theta) = \sum_\lambda \mathbf{R}H_\lambda$, $\lambda \in <\Theta>^+$, where $H_\lambda \in \mathfrak{a}$ is the coroot associated to λ, and let $\mathfrak{a}_\Theta$ be the orthogonal complement of $\mathfrak{a}(\Theta)$ in $\mathfrak{a}$. Set $\mathfrak{p}_\Theta = \mathfrak{b} + \mathfrak{n}^-(\Theta)$ where $\mathfrak{b} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}^+$ is the Lie algebra of B. Then the space $\mathfrak{p}_\Theta$ is the normalizer of $\mathfrak{n}_\Theta^+$ in $\mathfrak{g}$. Hence $\mathfrak{p}_\Theta$ is a Lie subalgebra of $\mathfrak{g}$ and the corresponding analytic subgroup P_Θ of G is closed. We have that P_Θ is a parabolic subgroup of G, and the 2^l subgroups constructed in this way for each possible choice of Θ are the so-called standard parabolic subgroups of G (with respect to our choice of Iwasawa decomposition). It can be shown that P_Θ admits the Langlands decomposition $P_\Theta = M_\Theta(K)AN^+$ where $M_\Theta(K)$ is the K-centralizer of $\mathfrak{a}_\Theta$. For each $w \in W$ write $\Sigma_w^+ = w\Sigma^- \cap \Sigma^+$ and denote by W_u the set of those $w \in W$ such that $\Sigma_w^+ \cap <\Theta>^+ = \varnothing$. Now we have:

A.2 Theorem *The quotient manifold G/P_Θ is the disjoint union of the N^+-orbits $N^+w_u^{-1}P_\Theta$*

for $w_u \in W_u$. Moreover, for each $w_u \in W_u$, the map $U^+_{w_u^{-1}} \to N^+ w_u^{-1} P_\Theta$, $v \mapsto v w_u^{-1} P_\Theta$ is a diffeomorphism.

In [22] Theorems A.1 and A.2 are respectively accredited to Bruhat – Harish-Chandra and Borel – Kostant. Note that Theorem A.1 is the particular instance of Theorem A.2 corresponding to $\Theta = \varnothing$. Note also that Theorem A.2 gives a cell decompositions for G/P_Θ which turns it into a finite CW-complex.

Next let X be a Riemannian symmetric space with no Euclidean factors which is also assumed to be noncompact, since the isotropy representation of a compact symmetric space coincides with that of its noncompact dual. Then we can find G, K as above such that $X = G/K$, namely G is the connected component of the isometry group of X and K is the isotropy subgroup at some point. Now the isotropy representation of X at $1 \cdot K$ is equivalent to the adjoint representation of K on $\mathfrak{p}$. Fix once and for all $p \in \mathfrak{a}$. The isotropy subgroup $K_p = M_\Theta(K)$ where Θ consists of those simple restricted roots which vanish at p. Thus we can identify the K-orbit through p with $K/M_\Theta(K) = G/P_\Theta$.

Let $q \in \mathfrak{a}$ be a regular element and consider the distance function $L_q : Kp \to \mathbf{R}$. Then L_q is a Morse function whose critical points are precisely the Weyl translates wp for $w \in W$. Therefore the Bott-Samelson cycles associated to L_q can be parametrized by W/W_p. For a given $w \in W$, consider the segment joining q and wp and suppose that this segment cuts across the singular hyperplanes corresponding to the reduced restricted roots $\lambda_1, \ldots, \lambda_k$ in this order. Then it is clear that w and $s_k \cdots s_1$ are W_p-conjugate where s_i is the reflection in the singular hyperplane $\{H \in \mathfrak{a} : \lambda_i(H) = 0\}$; let K_i denote the closed subgroup of K which is the stabilizer of that singular hyperplane. Denote the coset wW_p by $\bar{w}$. The Bott-Samelson cycle associated to $\bar{w}$ may be assumed to be $h_{\bar{w}} : \Gamma_{\bar{w}} \to K/K_p$, where $\Gamma_{\bar{w}} = K_1 \times_M K_2 \times_M \ldots \times K_k/M$ and $h_{\bar{w}}[(k_1, \ldots, k_k)] = k_1 \cdots k_k s_k \cdots s_1 K_p$. Let $\gamma_{\bar{w}}$ be a fundamental $\mathbf{Z}_2$-cycle in $\Gamma_{\bar{w}}$.

A.3 Theorem (Bott – Samelson) $\{(h_{\bar{w}})_*[\gamma_{\bar{w}}] : \bar{w} \in W/W_p\}$ *is a basis for the homology $H_*(K/K_p)$.*

The relation between Theorems A.2 and A.3 is given in the following theorem. The particular case when G is a complex Lie group and the K-orbit is a principal orbit was already treated by Hansen in [8].

A.4 Theorem *The subset $W_u^{-1} \subset W$ contains precisely one representative from each W_p-coset in W. For each $w \in W$, let $s_1, \ldots, s_k$ be as above and define $w_u = s_1 \cdots s_k$ so that w and w_u^{-1} are W_p-conjugate. Then $w_u \in W_u$ and the image $h_{\bar{w}}(\Gamma_{\bar{w}}) = K_1 \ldots K_k s_k \ldots s_1 K_p$ of the Bott-Samelson cycle $h_{\bar{w}} : \Gamma_{\bar{w}} \to K/K_p$ is the closure of the Bruhat cell $N^+ w_u^{-1} P_\Theta$.*

Proof. A word about notation: if $\lambda \in \Sigma_r$ is a reduced restricted root, we write $\bar{\mathfrak{g}}_\lambda = \mathfrak{g}_\lambda + \mathfrak{g}_{2\lambda}$. (Of course it could happen that 2λ is not a restricted root in which case $\mathfrak{g}_{2\lambda}$ would not be present.)

To see that $w_u \in W_u$ we need to verify that $\Sigma^+_{w_u} \cap \langle\Theta\rangle = \varnothing$. Since the reduced restricted roots in $w_u^{-1}\Sigma^+ \cap \Sigma^-$ are precisely $-\lambda_1, \ldots, -\lambda_k$, that statement is equivalent to having $\{-\lambda_1^{w_u}, \ldots, -\lambda_k^{w_u}\} \cap \langle\Theta\rangle = \varnothing$, which follows from the fact that wp does not belong to $\ker \lambda_i$ for $i = 1, \ldots, k$. Thus each W_p-coset in W contains at least one element of W_u^{-1}. We

invoke Proposition 1.1.2.13 in [22] to see that the number of W_p-cosets in W is equal to the cardinality of W_u^{-1}. This implies that each W_p-coset contains precisely one element of W_u^{-1}.

Note that $U^+_{w_u^{-1}}$ is a nilpotent Lie group with Lie algebra $\mathfrak{u}^+_{w_u^{-1}} = \sum_{i=1}^k \bar{\mathfrak{g}}_{\lambda_i}$ since the restricted roots in $\Sigma^+_{w_u^{-1}}$ are $\lambda_1, \ldots, \lambda_k$. Let U_i, $i = 1, \ldots, k$, denote the analytic subgroup of G with Lie algebra $\bar{\mathfrak{g}}_{\lambda_i}$. Then U_i is closed (because it is the connected component of the N^+-centralizer of the Lie subalgebra $\ker \lambda_i$ of $\mathfrak{a}$) and $U^+_{w_u^{-1}} = U_1 \cdots U_k$ (because $U^+_{w_u^{-1}}$ is nilpotent), so Theorem A.2 implies that $N^+ w_u^{-1} P_\Theta = U_1 \ldots U_k w_u^{-1} P_\Theta$ and that the dimension of $N^+ w_u^{-1} P_\Theta$ is equal to $\sum_{i=1}^k \dim(\bar{\mathfrak{g}}_{\lambda_i})$.

For each $j = 1, \ldots, k$ define $w_j = s_j \cdots s_2 s_1 \in W$. Set $B_j = w_j B w_j^{-1}$. Then B_j is a Borel subgroup of G with Lie algebra $\mathfrak{b}_j = \mathfrak{m} + \mathfrak{a} + \sum_{i=1}^j \bar{\mathfrak{g}}_{-\lambda_i} + \sum_{i=j+1}^k \bar{\mathfrak{g}}_{\lambda_i}$. Let also P_j be the parabolic subgroup of G with Lie algebra $\mathfrak{b}_j + \bar{\mathfrak{g}}_{\lambda_j}$. Then we have the inclusions between groups

$$P_1 \supset B_1 \subset P_2 \supset B_2 \subset P_3 \supset \ldots \subset P_k \supset B_k,$$

and we can form the quotient manifold $\mathcal{M}_{w_u} = P_1 \times_{B_1} \times P_2 \times_{B_2} \cdots \times P_k / B_k$.

Since $K_j \subset P_j$ (because $P_j = K_j A w_j N^+ w_j^{-1}$) and $M \subset B_j$ (because $B_j = MA w_j N^+ w_j^{-1}$), there is a map induced by inclusion $i : \Gamma_{\bar{w}} \to \mathcal{M}_{w_u}$ which is immediately seen to be injective and regular at $[(1, \ldots, 1)]$ (since $K_j \cap B_j = M$). By equivariance i is regular everywhere and it follows from the compactness of $\Gamma_{\bar{w}}$ and the fact that $\Gamma_{\bar{w}}$ and $\mathcal{M}_{w_u}$ have the same dimension that i is a diffeomorphism. Hence we have the following commutative diagram

$$\begin{array}{ccc} \Gamma_{\bar{w}} & \xrightarrow{\approx} & \mathcal{M}_{w_u} \\ h_{\bar{w}} \downarrow & & f_{w_u} \downarrow \\ K/K_p & \xrightarrow{\approx} & G/P_\Theta \end{array}$$

where $f_{w_u}[(p_1, \ldots, p_k)] = p_1 \ldots p_k s_k \ldots s_1 P_\Theta$. Now $h_{\bar{w}}(\Gamma_{\bar{w}}) = f_{w_u}(\mathcal{M}_{w_u}) = P_1 \ldots P_k s_k \ldots s_1 P_\Theta$ is compact and, as $P_j \supset U_j$, contains $U_1 \ldots U_k s_k \ldots s_1 P_\Theta = N^+ w_u^{-1} P_\Theta$. It follows that $h_{\bar{w}}(\Gamma_{\bar{w}})$ contains also the closure of $N^+ w_u^{-1} P_\Theta$.

We shall show next that we can write $P_j = U_j B_j \cup s_j B_j$ (disjoint union) where $U_j B_j$ is an open dense submanifold and $s_j B_j$ is a closed lower dimensional submanifold. For this purpose set $U_{-j} = \theta(U_j)$ and note that the G-centralizer G_j of $\ker \lambda_j$ is a reductive Lie group, $\ker \lambda_j$ being a θ-stable Abelian subalgebra of $\mathfrak{g}$. By the version of Theorem A.1 for reductive groups we can write $G_j = U_{-j} s_j MAU_{-j} \cup U_{-j} MAU_{-j} = U_{-j} s_j MAU_{-j} \cup MAU_{-j}$, since M and A normalize U_{-j}. Now multiply through by s_j; we have $s_j U_{-j} s_j = U_j$ and so it follows that $G_j = U_j MAU_{-j} \cup s_j MAU_{-j}$. As we know, $K_j \subset G_j$ and K_j acts transitively on P_j/B_j (with isotropy M). A fortiori, G_j acts transitively on P_j/B_j. Hence, $P_j = G_j B_j = U_j B_j \cup s_j B_j$ as claimed.

Owing to the above decomposition of P_j, by means of a straightforward induction argument one can easily establish that:

(i) Any element in $\mathcal{M}_{w_u}$ can be represented by an element $(v_1, \ldots, v_k) \in P_1 \times \ldots \times P_k$ where $v_i \in U_i \cup \{s_i\}$.

(ii) The image of $U_1 \times \ldots \times U_k$ in $\mathcal{M}_{w_u}$ is open and dense.

Since f_{w_u} is a diffeomorphism from $[U_1 \times \ldots \times U_k]$ onto the Bruhat cell $N^+ w_u^{-1} P_\Theta$, it finally follows from (i) and (ii) that $h_{\bar{w}}(\Gamma_{\bar{w}})$ equals the closure of $N^+ w_u^{-1} P_\Theta$. $\square$

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Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão, 1010
São Paulo, SP 05508-900
Brazil

E-mail: `gorodski@ime.usp.br`

Mathematisches Institut
Universität zu Köln
Weyertal 86-90
50931 Köln
Germany

E-mail: `gthorber@mi.uni-koeln.de`